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Decoherence induced by squeezing control errors in optical and ion trap holonomic quantum computations

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We study decoherence induced by stochastic squeezing control errors considering the particular implementation of Hadamard gate on optical and ion trap holonomic quantum computers. We analytically obtain both the purity of the final state and the fidelity for Hadamard gate when the control noise is modeled by Ornstein-Uhlenbeck stochastic process. We demonstrate the purity and the fidelity oscillations depending on the choice of the initial superimposed state. We derive a linear formulae connecting the gate fidelity and the purity of the final state.

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I. INTRODUCTION

Holonomic quantum computations exploiting non-abelian geometrical phases [1] was primarily proposed in the Ref. [2] and developed further in the Ref. [3]. Many implementations of holonomic quantum computers (HQC) have been proposed. Particularly, the realization of HQC within quantum optics was suggested (optical HQC) [4]. Laser beams in a non-linear Kerr medium were exploited for this purpose. Two different sets of control devices can be used in this case. The first one considered in this paper consists of one- and two-mode displacing and squeezing devices. The second one includes SU(2) interferometers. As well trapped ions with the excited state connected to a triple degenerate subspace (four level Λ-system) can be used to implement HQC [5]. Another approach to HQC exploiting squeezing and displacement of the trapped ions vibrational modes was suggested in the Ref. [6]. This implementation of HQC is mathematically similar to the first embodiment of the optical HQC [4] and thus it is also considered in this work. Particularly, expressions for the adiabatic connection and holonomies are the same in these cases. Another proposed implementation of HQC was the HQC with neutral atoms in cavity QED [7]. The coding space was spanned by the dark states of the atom trapped in a cavity. Dynamics of the atom was governed by the generalized Λ-system Hamiltonian. Mathematically similar semiconductor-based implementation of HQC was proposed in the Refs. [8], where one-qubit gates were also realized in the framework of the generalized Λ-system. In distinction from the cavity model of HQC its physical implementation exploits semiconductor excitons driven by sequences of laser pulses [8]. For the two-qubit gate implementation the bi-excitonic shift was used. The generalized Λ-system with the different Rabi frequencies parametrization was exploited recently for HQC implemented by Rf-SQUIDs coupled through a microwave cavity [9]. One more solid state implementation of HQC based on Stark effect was proposed in the Ref. [10].

Let us briefly remind the main results concerning the holonomic quantum computation. In HQC non-abelian geometric phases (holonomies) are exploited to implement unitary transformations over the quantum code. The later is some degenerate subspace C^N spanned on eigenvectors of Hamiltonian H_0, which initiates the parametric isospectral family of Hamiltonians $F = \{H(\lambda) = U(\lambda)H_0U^\dagger(\lambda)\}_{\lambda\in M}$. Here $U(\lambda)$ is a unitary operator, λ is a vector belonging to the space of the control parameters M and N denotes the dimension of the degenerate computational subspace [2, 3]. Quantum gates are implemented when the control parameters are adiabatically driven along the loops in the control manifold M. The unitary operator mapping the initial state vector belonging to C^N into the final one has the form $e^{i\phi}\Gamma_\gamma(A_\mu)$, where the index μ enumerates control parameters, λ_μ constitute vector λ and ϕ is the dynamical phase. Holonomy associated with the loop $\gamma \in M$ is

$$\Gamma_\gamma(A_\mu) = \hat{\mathbf{P}} \exp\left\{\int_\gamma A_\mu d\lambda_\mu\right\}. \tag{1}$$

Here $\hat{\mathbf{P}}$ denotes the path ordering operator, A_μ is the matrix valued adiabatic connection given by the expression [1]:

$$(A_\mu)_{mn} = \langle\varphi_m| U^\dagger \frac{\partial}{\partial\lambda_\mu} U |\varphi_n\rangle, \tag{2}$$

where $|\varphi_k\rangle$ with $k = \overline{1,N}$ are the eigenvectors of the Hamiltonian H_0 forming the basis in C^N. Dynamical phase ϕ will be omitted bellow due to the suitable choice of the zero energy level. We shall consider the single subspace C^N (no energy level crossings are assumed).

It is evident that the quantum gate (holonomy) performed depends on the path passed in the control parameters space. As well it is obvious that in real experiments it is impossible to pass the desired loop in the control manifold without any deviations. Errors in the assign-

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ment of the classical control parameters λ are unavoidable. The question about robustness of holonomic quantum computations with respect to the control errors has attracted a lot of attention recently. Namely, the effect of the errors originated from the imperfect control of classical parameters was studied for $\mathbf{CP}^n$ model of HQC in the Ref. [11] where the control-not and Hadamard gates were particularly considered. Berry phase for the spin 1/2 particle in a classical fluctuating magnetic field was considered in the Ref. [12]. Approach based on the non-abelian Stokes theorem [13] was proposed in the Ref. [14]. Namely, the general expression for the fidelity valid for arbitrary implementation of HQC in the case of the single control error having arbitrary size and duration was derived. Simple approximate formulae was obtained in the small error limit. Adiabatic dynamics of quantum system coupled to a noisy classical control field was studied in the Ref. [15]. It was demonstrated that stochastic phase shift arising in the off-diagonal elements of the system's density matrix can cause decoherence. The efficiency of Shor algorithm [16] run on a geometric quantum computer was investigated in the case when the decoherence induced by the stochastic control errors was taken into account. The study of the robustness of the non-abelian holonomic quantum gates with respect to the stochastic fluctuations of the control parameters was presented in the Ref. [17]. Three stability regimes were discriminated in this work for the HQC model with qubits given by polarized excitonic states controlled by laser pulses. Noise cancellation effect for simple quantum systems was considered in the Ref. [18]. Robustness of the parametric family of quantum gates subjected to stochastic fluctuations of the control parameters was studied in the Ref. [19]. Usage of the cyclic states [20] allowed to consider quantum gates which could be continuously changed from dynamic gates to purely geometric ones. It was shown that the maximum of the gate fidelity corresponds to quantum gates with a vanishing dynamical phase. Robust Hadamard gate implementation for optical [4] and ion trap [6] holonomic quantum computers was proposed in the Ref. [21]. The cancellation of the small squeezing control errors up to the fourth order on their magnitude was demonstrated. Hadamard gate is one of the key elements of the main quantum algorithms, for instance see [16, 22]. Thus the search for its robust implementations is of importance.

During the last few years much attention has been payed to the study of both abelian and non-abelian geometric phases in the presence of decoherence which is the most important limiting factor for quantum computations. Let us briefly overview some of these works. The abelian geometric phase of the two-level quantum system interacting with a one and two mode quantum field subjected to the decoherence was considered in the Ref. [23]. It was demonstrated that when the geometric phase is generated by an adiabatic evolution the first correction due to the decoherence of the driving quantized field for the no-jump trajectory has the second order in the decaying rate of the field but it is not the case for the non-adiabatic evolution. Non-abelian holonomies in the presence of decoherence were investigated in the Ref. [24] using the quantum jump approach. The effects of environment on a universal set of holonomic quantum gates were analyzed. Refocusing schemes for holonomic quantum computation in the presence of dissipation were discussed in the Ref. [25]. It has been shown that non-abelian geometric gates realized by means of refocused double-loop scheme possessed a certain resilience against decoherence. Quantum Langevin approach has been used to study the evolution of two-level system with a slowly varying Hamiltonian and interacting with a quantum environment modeled as a bath of harmonic oscillators [26]. It allowed to obtain the dissipation time and the correction to Berry phase in the case of adiabatic cyclic evolution. The realization of universal set of holonomic quantum gates acting on decoherence-free subspaces has been proposed in the Ref. [27]. It has been shown how it can be implemented in the contexts of trapped ions and quantum dots. The performance of holonomic quantum gates in semi-conductor quantum dots under the effect of dissipative environment has been studied in the Ref. [28]. It was demonstrated that the influence of the environment modeled by the superhomic thermal bath of harmonic oscillators could be practically suppressed. The study of the non-adiabatic dynamics and effects of quantum noise for the ion trap setup proposed in the Ref. [5] has been also done [29]. The optimal finite operation time was determined. In the references mentioned above the fidelity was used as the main measure of gate resilience.

In this paper we consider optical and ion trap implementations of HQC proposed in the Refs. [4] and [6] respectively. Regarding the particular implementation of Hadamard gate we study the decoherence induced by stochastic squeezing control errors. Following the Ref. [12] we model the random fluctuations by Ornstein-Uhlenbeck stochastic process. We analytically obtain the final state purity and the gate fidelity as the measures of the gate robustness with respect to the decoherence induced by stochastic control errors. In the small squeezing control errors limit we derive a simple formulae connecting the purity of the final state and the gate fidelity.

II. HADAMARD GATE IMPLEMENTATION

Optical and ion trap setups of HQC are mathematically equivalent since the corresponding holonomies are the same (compare the Refs. [4] and [6]). Therefore we can consider both HQC models simultaneously.

The laser beams in the nonlinear Kerr medium are explored in order to perform holonomic quantum computation in the framework of the optical setup. The corresponding interaction Hamiltonian describing a single beam in the medium is

$$H_I = \hbar X a^\dagger a \left(a^\dagger a - 1\right), \tag{3}$$

where a and $a^\dagger$ are the annihilation and creation opera-

tors of the photons respectively, X is a constant proportional to the third order nonlinear susceptibility of the medium. The degenerate computational subspace of the single qubit is spanned on the photon Fock states $|0\rangle$ and $|1\rangle$. More details one can find in the Ref. [4].

In order to implement the same holonomic quantum computational scheme in the framework of the ion trap setup one has to deal with the two-level trapped ion placed in the common node of two standing electromagnetic waves with frequencies $\omega_0-\omega_z$ and $\omega_0+\omega_z$ as well as being affected by the traveling wave with the frequency ω_0. Here ω_0 denotes the frequency corresponding to the transition between the two ion levels being exploited, ω_z is the frequency of the ion's harmonic oscillations along the z axis of the linear Paul trap. The basis qubit states are $|g\rangle\otimes|0\rangle$ and $\left(|g\rangle\otimes|1\rangle-|e\rangle\otimes|0\rangle\right)/\sqrt{2}$. Here $|g\rangle$ and $|e\rangle$ are the ground and excited internal states of the ion, $|0\rangle$ and $|1\rangle$ are the two lowest vibrational Fock states of the ion in the trap. More details can be found in the Ref. [6].

One-qubit gates are given as a sequence of single mode squeezing and displacing operations [4, 6]:

$$U(\eta,\nu)=D(\eta)S(\nu), \tag{4}$$

where

$$\begin{aligned} S(\nu) &= \exp\left(\nu a^{\dagger 2}-\nu^* a^2\right), \\ D(\eta) &= \exp\left(\eta a^\dagger-\eta^* a\right) \end{aligned} \tag{5}$$

denote single mode squeezing and displacing operators respectively, $\nu=r_1 e^{i\theta_1}$ and $\eta=x+iy$ are corresponding complex control parameters, a and $a^\dagger$ are annihilation and creation operators. The asterix denotes complex conjugation. The expressions for the adiabatic connection and the curvature tensor can be found in the Refs. [4, 6]. Following our previous Letter [21] we consider Hadamard gate

$$H_0=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \tag{6}$$

implemented when two rectangular loops belonging to the planes $(x,r_1)\,|_{\theta_1=0}$ and $(y,r_1)\,|_{\theta_1=0}$ are passed. Namely,

$$-iH_0=\Gamma(C_{II})\left|_{\Sigma_{II}=\pi/2}\Gamma(C_I)\right|_{\Sigma_I=\pi/4}, \tag{7}$$

where the holonomies are

$$\begin{aligned} \Gamma(C_I) &= \exp\left(-i\sigma_y\Sigma_I\right), \quad \Sigma_I=\int\limits_{S(C_I)} dx dr_1 2e^{-2r_1}, \\ \Gamma(C_{II}) &= \exp\left(-i\sigma_x\Sigma_{II}\right), \quad \Sigma_{II}=\int\limits_{S(C_{II})} dy dr_1 2e^{2r_1}, \end{aligned} \tag{8}$$

and $S(C_{I,II})$ are the regions in the planes $(x,r_1)\,|_{\theta_1=0}$ and $(y,r_1)\,|_{\theta_1=0}$ enclosed by the rectangular loops C_I and C_{II} respectively. The sides of the rectangles C_I and C_{II} are parallel to the coordinate axes. For the loop C_I these sides are given by the lines $r_1=0$, $x=b_x$, $r_1=d_x$ and $x=a_x$, where the length of the rectangle's sides parallel to the x axis is $l_x=b_x-a_x$. In the Ref. [21] it was shown that

$$d_x=-\frac{1}{2}\ln\left(1-\frac{\pi}{4l_x}\right), \quad l_x>\frac{\pi}{4}. \tag{9}$$

In the same way the rectangle C_{II} is composed of the lines $r_1=0$, $y=b_y$, $r_1=d_y$ and $y=a_y$, where [21]:

$$d_y=\frac{1}{2}\ln\left(1+\frac{\pi}{2l_y}\right). \quad l_y=b_y-a_y. \tag{10}$$

Proposed Hadamard gate implementation is not a unique one. The same gate can be realized by passing another loops in the control manifold. Our choice is motivated by the simplicity of the loops.

III. DECOHERENCE INDUCED BY STOCHASTIC SQUEEZING CONTROL ERRORS

We restrict ourselves by the consideration of the squeezing control errors only. Moreover, we can neglect the fluctuations of the squeezing control parameter when $r_1=0$. Thus to take into account random squeezing control errors we have to replace d_x by $d_x+\delta r_x(x)$ and d_y by $d_y+\delta r_y(y)$, where $\delta r_x(x)$ and $\delta r_y(y)$ are two independent Ornstein-Uhlenbeck stochastic processes. Making this substitution into the Eqs. (8) instead of the formulae (7) we obtain the following expression for the perturbed Hadamard gate, see also [21]:

$$\begin{aligned} -iH = &-\frac{1}{\sqrt{2}}\left(\cos\alpha-\sin\alpha\right)\left(\sin\beta+i\sigma_x\cos\beta\right)- \\ &-\frac{i}{\sqrt{2}}\left(\cos\alpha+\sin\alpha\right)\left(\sigma_z\cos\beta-\sigma_y\sin\beta\right), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \alpha &= e^{-2d_x}\int\limits_{a_x}^{b_x} dx\left(1-e^{-2\delta r_x}\right), \\ \beta &= e^{2d_y}\int\limits_{a_y}^{b_y} dy\left(e^{2\delta r_y}-1\right). \end{aligned} \tag{12}$$

Let the qubit initially to be in the pure superimposed state $|\psi_0\rangle=c_0\,|0\rangle+c_1\,|1\rangle$, where amplitudes c_0 and c_1 obey the normalization constrain $|c_0|^2+|c_1|^2=1$. For the fixed noise realization the final qubit state will be pure. However, it will differ from the desired one. In the real experiment we do not follow the random fluctuations of the control parameters (nevertheless *in principle* we can do it). In this situation quantum mechanics prescribes us to describe the final state of the system by the

density matrix and represent the state as a mixture of all possible final states weighted with the probabilities of the corresponding noise realizations. Following this strategy we find the density matrix of the final state for a given noise implementation and than average over the squeezing control parameter fluctuations when the later are modeled by the two independent Ornstein-Uhlenbeck stochastic processes.

Thus, for the density operator $\rho \equiv H \left|\psi_0\right\rangle \left\langle\psi_0\right| H^\dagger$ we obtain the following matrix elements:

$$\begin{aligned}
\langle 0|\rho|0\rangle &= \frac{1}{2} + \frac{1}{2}\left(|c_0|^2 - |c_1|^2\right)\cos 2\beta\cos 2\gamma - \frac{1}{2}\left(c_0 c_1^* + c_0^* c_1\right)\cos 2\beta \sin 2\gamma - \frac{i}{2}\left(c_0 c_1^* - c_0^* c_1\right)\sin 2\beta, \\
\langle 0|\rho|1\rangle &= \frac{1}{2}\left(|c_1|^2 - |c_0|^2\right)\sin 2\gamma + \frac{i}{2}\left(|c_0|^2 - |c_1|^2\right)\sin 2\beta\cos 2\gamma - \frac{1}{2}\left(c_0 c_1^* + c_0^* c_1\right)\left(\cos 2\gamma + i \sin 2\beta \sin 2\gamma\right) \\
&\quad - \frac{1}{2}\left(c_0 c_1^* - c_0^* c_1\right)\cos 2\beta, \\
\langle 1|\rho|0\rangle &= \langle 0|\rho|1\rangle^*, \quad \langle 1|\rho|1\rangle = 1 - \langle 0|\rho|0\rangle.
\end{aligned} \tag{13}$$

Here the introduced parameter γ is defined as $\gamma = \alpha - \pi/4$ and the asterix denotes the complex conjugate quantities.

We assume that the noise δr_x has variance $\tilde{\sigma}_x$ and a lorentzian spectrum with the bandwidth Γ_x. The fluctuations δr_y have the variance $\tilde{\sigma}_y$ and bandwidth Γ_y. Using the Eqs. (12)-(13) and the properties of Ornstein-Uhlenbeck stochastic process (see Ref. [30]):

$$\begin{aligned}
\overline{\delta r_{x,y}(t_1)\delta r_{x,y}(t_2)} &= \tilde{\sigma}_{x,y} e^{-\Gamma_{x,y}|t_1 - t_2|}, \\
\overline{\delta r_{x,y}^2} = \tilde{\sigma}_{x,y}, &\quad \overline{\delta r_{x,y}} = 0,
\end{aligned} \tag{14}$$

we average the density matrix ρ over the stochastic fluctuations of the squeezing control parameters δr_x and δr_y. The line over the random quantities means the averaging operation. We assume that $\delta r_{x,y} \ll 1$ and restricted ourselves by the first non-vanishing terms depending on δr_x or δr_y. As the result of straightforward but a bit lengthy calculations we analytically obtain the elements of the averaged density matrix $\overline{\rho}$:

$$\begin{aligned}
\langle 0|\overline{\rho}|0\rangle &= \frac{1}{2}\left(1 + c_0 c_1^* + c_0^* c_1\right) - 2\left(|c_0|^2 - |c_1|^2\right)\tilde{\sigma}_x l_x e^{-2d_x} - 2i\left(c_0 c_1^* - c_0^* c_1\right)\tilde{\sigma}_y l_y e^{2d_y} \\
&\quad - \left(c_0 c_1^* + c_0^* c_1\right)\left[\frac{8\tilde{\sigma}_x}{\Gamma_x}\mathcal{F}_x + \frac{8\tilde{\sigma}_y}{\Gamma_y}\mathcal{F}_y\right], \\
\langle 0|\overline{\rho}|1\rangle &= \frac{1}{2}\left(c_0 + c_1\right)\left(c_0^* - c_1^*\right) + 2\left(c_0 c_1^* + c_0^* c_1\right)\left(\tilde{\sigma}_x l_x e^{-2d_x} + i\tilde{\sigma}_y l_y e^{2d_y}\right) \\
&\quad - \frac{8\tilde{\sigma}_x}{\Gamma_x}\left(|c_0|^2 - |c_1|^2\right)\mathcal{F}_x + \frac{8\tilde{\sigma}_y}{\Gamma_y}\left(c_0 c_1^* - c_0^* c_1\right)\mathcal{F}_y, \\
\langle 1|\overline{\rho}|0\rangle &= \langle 0|\overline{\rho}|1\rangle^*, \quad \langle 1|\overline{\rho}|1\rangle = 1 - \langle 0|\overline{\rho}|0\rangle.
\end{aligned} \tag{15}$$

Here we introduced the following denotations:

$$\begin{aligned}
\mathcal{F}_x &= e^{-4d_x}\left(l_x - \frac{1 - e^{-\Gamma_x l_x}}{\Gamma_x}\right), \\
\mathcal{F}_y &= e^{4d_y}\left(l_y - \frac{1 - e^{-\Gamma_y l_y}}{\Gamma_y}\right).
\end{aligned} \tag{16}$$

In order to quantify decoherence strength we exploit the purity of the final state. It is defined as the trace of the squared density matrix. Purity equals to 1 for pure states and less than 1 overwise. Using the Eqs. (13) it is easy to check that for a fixed noise realization the purity $I_0 = tr\rho^2$ equals the unity as it should be. Using the Eqs. (15) we obtain the purity of the final state in the case of the stochastic squeezing control errors. The result of the lengthy but straightforward calculations is

$$\begin{aligned}
I = tr\overline{\rho}^2 = 1 &- \frac{64\tilde{\sigma}_y}{\Gamma_y}\mathcal{F}_y |c_0|^2 |c_1|^2 \\
&- \frac{16\tilde{\sigma}_x}{\Gamma_x}\mathcal{F}_x\left(c_0^2 + c_1^2\right)\left(c_0^{*2} + c_1^{*2}\right),
\end{aligned} \tag{17}$$

Thus we see that the final state purity $I < 1$ and the stochastic squeezing control errors induce decoherence and lead the final state to be a mixture of pure states.

We can simplify the expression (17) if we exploit the following parametrization for the amplitudes c_0 and c_1:

$$c_0 = e^{i\xi}\cos\varphi, \quad c_1 = e^{i\chi}\sin\varphi, \tag{18}$$

and assume that $\xi - \chi = \pi n$, where n is an integer. In this rather general case formulae (17) reduces to

$$I = 1 - \frac{16\tilde{\sigma}_x}{\Gamma_x}\mathcal{F}_x - \frac{16\tilde{\sigma}_y}{\Gamma_y}\mathcal{F}_y \sin^2 2\varphi. \tag{19}$$

We see that the final state purity oscillates depending on the choice of the initial qubit state. The purity decreases when the initial qubit state is a superposition of the basis states. For example, it has its minimum when $|\psi_0\rangle = \left(|0\rangle \pm |1\rangle\right)/\sqrt{2}$. In contrary, the purity has its maximum if the initial qubit state $|\psi_0\rangle$ is proportional to one of the basis states $|0\rangle$ or $|1\rangle$. In this case the errors made in the $(y, r_1)\,|_{\theta_1=0}$ plane are eliminated.

IV. HADAMARD GATE FIDELITY

Now we obtain the fidelity of the non-ideal Hadamard gate. In the case when there are no control errors ($\delta r_x = \delta r_y = 0$) the density matrix $\rho^{(0)}$ of the final (pure) state has the following form:

$$\rho^{(0)} \equiv H_0 |\psi_0\rangle \langle\psi_0| H_0 = \frac{1}{2}\begin{pmatrix} 1 + c_0^* c_1 + c_0 c_1^* & |c_0|^2 - |c_1|^2 + c_0^* c_1 - c_0 c_1^* \\ |c_0|^2 - |c_1|^2 + c_0 c_1^* - c_0^* c_1 & 1 - c_0^* c_1 - c_0 c_1^* \end{pmatrix}. \tag{20}$$

The non-ideal Hadamard gate fidelity $F \equiv tr(\rho^{(0)}\overline{\rho})$ under the same assumptions as in the Eq. (15) is given by the expression

$$F = 1 - \frac{32\tilde{\sigma}_y}{\Gamma_y}\mathcal{F}_y |c_0|^2 |c_1|^2 - \frac{8\tilde{\sigma}_x}{\Gamma_x}\mathcal{F}_x \left(c_0^2 + c_1^2\right)\left(c_0^{*2} + c_1^{*2}\right). \tag{21}$$

From this expression we can conclude that when the initial state vector is proportional either to $|0\rangle$ or $|1\rangle$ the contribution of the control errors made in the $(y, r_1)\,|_{\theta_1=0}$ plane can be neglected at the accepted approximation degree. In our previous work [21] the fidelity was defined as $f = \sqrt{F}$ and the condition $|c_0|^2 |c_1|^2 = 0$ was assumed. In the limit $(\Gamma_x l_x)^{-1} \to 0$, when the fluctuations average out, from the Eq. (21) we obtain that $1 - f \sim \tilde{\sigma}_x^2$. Thus our previous result [21] concerning the cancellation of the squeezing control errors up to the fourth order on their magnitude is reproduced as it should be (remind that $\tilde{\sigma}_x$ has the order of $(\delta r_x)^2$).

We can simplify the expression (21) exploiting the parametrization (18) for the amplitudes c_0 and c_1 and assuming $\xi - \chi = \pi n$, where n is an integer. Under these assumptions the gate fidelity is given by

$$F = 1 - \frac{8\tilde{\sigma}_x}{\Gamma_x}\mathcal{F}_x - \frac{8\tilde{\sigma}_y}{\Gamma_y}\mathcal{F}_y \sin^2 2\varphi. \tag{22}$$

It is evident that the gate fidelity oscillates depending on the choice of the initial qubit state. It is less for the superimposed initial states than for the basis ones. Namely, the fidelity has its maximum when the initial qubit state $|\psi_0\rangle$ is proportional either to $|0\rangle$ or $|1\rangle$. In this case the errors made in the $(y, r_1)\,|_{\theta_1=0}$ plane are eliminated. As well the fidelity has its minimum when the initial state vector is equal to $\left(|0\rangle \pm |1\rangle\right)/\sqrt{2}$. Moreover from Eqs. (17) and (21) we find a simple linear formulae connecting the purity of the final state and the gate fidelity:

$$I = 2F - 1. \tag{23}$$

This expression demonstrates the close connection between these quantities defining the decoherence strength and the gate stability.

V. CONCLUSIONS

We considered optical and ion trap HQC proposed the Refs. [4] and [6] respectively. Regarding the particular implementation of Hadamard gate we have studied decoherence induced by stochastic squeezing control errors. Ornstein-Uhlenbeck stochastic process was exploited to model random fluctuations of the squeezing control parameter. We have analytically obtained the purity of the final qubit state and calculated the fidelity of the non-ideal Hadamard gate. It was shown that the stochastic squeezing control errors reduce the final state into a mixture of pure states and, thus, induce decoherence. We have shown that the final state purity oscillates depending on the choice of the initial qubit state. The purity decreases when the initial qubit state is a superposition of the basis states. For example, it has its minimum when the initial state vector is equal to $\left(|0\rangle \pm |1\rangle\right)/\sqrt{2}$. In contrary, the purity has its maximum when the initial qubit state is proportional to one of the basis states $|0\rangle$ or $|1\rangle$. In this case the control errors made in the $(y, r_1)\,|_{\theta_1=0}$ plane are eliminated. The same conclusions can be made for the gate fidelity. Simple linear formulae connecting the gate fidelity and the purity of the final state was derived.

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